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The solution of some quantum nonlinear oscillators with the common symmetry group $SL(2, R)$

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Abstract

In a series of papers Calogero and Graffi (2003 *Phys. Lett.* A **313** 356–62) and Calogero (*Phys. Lett.* A (submitted); *J. Nonlinear Math. Phys.* (to appear)) treated the quantization of several one-degree-of-freedom Hamiltonians containing a parameter, c, which plays no role in the classical motion, but is critical to the value of the eigenvalue of the ground state. In this paper we examine the classical and quantum problems from the point of view of their Noether and Lie point symmetries respectively and demonstrate the construction of the quantal wavefunctions from the Lie point symmetries of the Schrödinger equation.

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1. Introduction

In a series of papers Calogero and Graffi [2] and Calogero [3, 4] discussed the quantization of a selection of one-degree-of-freedom Hamiltonians containing a parameter c which plays no role in the classical motion. However, it does play an important role in the quantized problem. Moreover it was observed that the spectra obtained for different systems which are related by a nonlinear canonical transformation differed. Of some interest was the fact that the classical problems were all isochronous with period 2π. We term such systems c-isochronous nonlinear oscillators. Of the systems considered by Calogero the two which are related by a nonlinear canonical transformation are of particular interest. They are [2]

$$H_1(p,q) = \frac{1}{2}\left[\frac{p^2q^3}{c} + c\left(q + \frac{1}{q}\right)\right] \tag{1.1}$$

and [4]

$$H^{(s)}(p,q) = \frac{1}{2}\left[\frac{p^2q}{c} + c\left(q + \frac{s}{q}\right)\right], \tag{1.2}$$

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where[2] $s = \pm 1$. Calogero takes the parameter c to be an arbitrary real positive constant. This is reasonable if one wants to maintain the physical interpretation of $\hat{H}$ as a Hermitian operator. However, from a mathematical point of view one is not obliged to do so immediately and for the nonce we make no statement about the value of the parameter c, but wait until the mathematics requires us to impose some constraint on its value.

Hamilton's equations of motion for (1.2) are

$$\dot{q} = \frac{qp}{c} \qquad \dot{p} = \frac{1}{2c}\left[p^2 + c^2\left(\frac{s}{q^2} - 1\right)\right] \tag{1.3}$$

from which we deduce the Newtonian equation

$$\ddot{q} = \frac{1}{2q}(\dot{q}^2 - q^2 + s) \tag{1.4}$$

in which it is evident that the parameter c is absent. If we proceed via the Lagrangian, we have

$$\begin{aligned} L &= p\dot{q} - H \\ &= \frac{1}{2}c\left[\frac{\dot{q}^2}{q} - \left(q + \frac{s}{q}\right)\right] \end{aligned} \tag{1.5}$$

in which we have used $p = c\dot{q}/q$ from (1.3a). The parameter c is a multiplier of the Lagrangian and so is classically irrelevant.

One observes that the foregoing discussion raises a question about the process of quantization. In addition to the standard problem of normal ordering of classical variables when they become quantal operators there appears to be some ambiguity in the whole procedure of quantization due to the possibility of the intrusion of a parameter, the constant c above, which has no relevance to the classical motion and yet impacts significantly on the quantal problem.

In this paper we wish to examine more closely the solution of the Schrödinger equation for the two operators (1.1) and (1.2). In particular we determine the values of the parameter c which are permissible. Our approach is based on symmetry and consequently follows the lines developed in Lemmer *et al* [9] in their treatment of the Schrödinger equations for the simple harmonic oscillator and the Ermakov–Pinney problem in one dimension. We see that the latter problem is closely related to the two Hamiltonians, (1.1) of Calogero and Graffi and (1.2) of Calogero.

Calogero [4] concludes with the observation that the results reported in his three papers are probably pedagogically useful and that perhaps they should be included in teaching a first course in quantum mechanics. We agree with that observation and would hope that some of the methods and results reported in this paper would be part of the material to be presented.

2. The classical symmetries

The Lie point symmetries of (1.4) are easily obtained with one of the symbolic manipulation codes available for the computation of symmetries of differential equations. We use LIE [8, 12] and obtain the three Lie point symmetries

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= \cos t\,\partial_t - q\sin t\,\partial_q \\ \Gamma_3 &= \sin t\,\partial_t + q\cos t\,\partial_q \end{aligned} \tag{2.1}$$

with the Lie brackets

$$[\Gamma_1, \Gamma_2]_{\mathrm{LB}} = -\Gamma_3, \qquad [\Gamma_2, \Gamma_3]_{\mathrm{LB}} = \Gamma_1, \qquad [\Gamma_3, \Gamma_1]_{\mathrm{LB}} = -\Gamma_2 \tag{2.2}$$

[2] Here we depart somewhat from the notation of Calogero [4]. In that paper Calogero writes, repeatedly, $s = \pm$.

so that the symmetries are a representation of the Lie algebra $so(2, 1)$, the noncompact form of rotations in three dimensions. We find it more convenient to rewrite the symmetries as

$$
\begin{aligned}
\Gamma_1 &= \mathrm{i}\partial_t \\
\Gamma_{2\pm} &= \Gamma_2 \pm \mathrm{i}\Gamma_3 \\
&= \mathrm{e}^{\pm \mathrm{i}t}[\partial_t \pm \mathrm{i}q\partial_q]
\end{aligned}
\tag{2.3}
$$

for which the Lie brackets are now

$$
[\Gamma_1, \Gamma_{2\pm}]_{\mathrm{LB}} = \mp\Gamma_{2\pm}, \qquad [\Gamma_{2+}, \Gamma_{2-}]_{\mathrm{LB}} = 2\Gamma_1 \tag{2.4}
$$

and we have a representation of $sl(2, R)$.

The Noether symmetries of the action integral associated with the Lagrangian (1.5) can be verified [1] by testing the Lie symmetries in the equation for the boundary term in Noether's theorem. In the process we determine the boundary term and also the associated integral. For the particular Lagrangian, (1.5), the general formulae [10] become

$$
\dot{f} = c\left\{-\dot{\tau}\frac{1}{2}\left[\frac{\dot{q}^2}{q} + \left(q + \frac{s}{q}\right)\right] - \eta\frac{1}{2}\left[\frac{\dot{q}^2}{q^2} + \left(1 - \frac{s}{q^2}\right)\right] + \dot{\eta}\frac{\dot{q}}{q}\right\} \tag{2.5}
$$

$$
I = f + \frac{1}{2}c\tau\left[\frac{\dot{q}^2}{q} + \left(q + \frac{s}{q}\right)\right] - c\eta\frac{\dot{q}}{q}. \tag{2.6}
$$

Specifically we find that

$$
\Gamma_1 : \quad f_1 = 0 \qquad I_1 = \frac{1}{2}c\left[\frac{\dot{q}^2}{q} + \left(q + \frac{s}{q}\right)\right] \tag{2.7}
$$

$$
\Gamma_{2\pm} : \quad f_{2\pm} = -cq\,\mathrm{e}^{\pm \mathrm{i}t} \qquad I_{2\pm} = \frac{1}{2}c\,\mathrm{e}^{\pm \mathrm{i}t}\left[\frac{\dot{q}^2}{q} \mp 2\mathrm{i}q - \left(q - \frac{s}{q}\right)\right]. \tag{2.8}
$$

All three Lie point symmetries of the Newtonian equation of motion (1.4) are Noether point symmetries of the action integral. The algebra of the Noether point symmetries is $sl(2, R)$ which is the same as one finds for the Ermakov–Pinney problem and so one expects that the construction of the solutions of the Schrödinger equation corresponding to the Hamiltonian (1.2) leads to similar results as for that problem.

3. Symmetries and wavefunctions for $H^{(s)}$

The time-dependent Schrödinger equation corresponding to the Hamiltonian (1.2) is, when one uses the ordering prescription of Calogero [4] so that there is consistency between his treatment and the present discussion,

$$
2\mathrm{i}c\frac{\partial u}{\partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} - c^2\left(x + \frac{s}{x}\right)u = 0 \tag{3.1}
$$

for which we have written $\hat{q} \Rightarrow x$ and $\hat{p} \Rightarrow -\mathrm{i}\partial_x$. Equation (3.1) is the time-dependent Schrödinger equation from which the time-independent form of Calogero [4] (17) is obtained by the standard method of separation of variables. With the aid of LIE and some rearrangement of the results we obtain the Lie point symmetries of (3.1) to be

$$
\begin{aligned}
\Gamma_1 &= \mathrm{i}\partial_t \\
\Gamma_{2\pm} &= \mathrm{e}^{\pm \mathrm{i}t}\left[\partial_t \pm \mathrm{i}x\partial_x - \left(\mathrm{i}cx \pm \tfrac{1}{2}\mathrm{i}\right)u\partial_u\right] \\
\Gamma_3 &= u\partial_u \\
\Gamma_f &= f(t, x)\partial_u,
\end{aligned}
\tag{3.2}
$$

where $f(t, x)$ is any solution of (3.1). The finite algebra is $A_1 \oplus_s sl(2, R)$ with the homogeneity symmetry Γ_3 providing the one-dimensional Abelian subalgebra and the $sl(2, R)$ of Γ_1 and $\Gamma_{2\pm}$ corresponding to the algebra of the classical Noether symmetries as one finds in the case of the simple harmonic oscillator and the Ermakov–Pinney problem [9]. The Lie bracket of any of the finite symmetries with Γ_f produces another representative of the class of symmetries denoted by Γ_f.

To determine the wavefunctions of the time-dependent Schrödinger equation (3.1) one looks for similarity solutions corresponding to the symmetries. For this purpose Γ_1 is of no use even though it is the symmetry which enables the standard separation of variables which leads to the time-independent Schrödinger equation. The homogeneity symmetry, Γ_4, and the solution symmetry, Γ_f, are of even less use. Fortunately the symmetries, $\Gamma_{2\pm}$, are useful. We calculate the invariants from the associated Lagrange system

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{\pm \mathrm{i}x} = \frac{\mathrm{d}u}{-\left(\mathrm{i}cx \pm \frac{1}{2}\mathrm{i}\right)u}. \tag{3.3}$$

From the first and second of (3.3) we obtain the invariant

$$v = x\,\mathrm{e}^{\mp \mathrm{i}t} \tag{3.4}$$

and from the second and third of (3.3) with the aid of (3.4) the invariant

$$w = u \exp\left[\pm\left(cx + \tfrac{1}{2}\mathrm{i}t\right)\right]. \tag{3.5}$$

We use Γ_{2+} to generate the solution of (3.1) so that the structure of the similarity solution being sought is

$$u = \exp\left[-\left(cx + \tfrac{1}{2}\mathrm{i}t\right)\right] f(x\,\mathrm{e}^{-\mathrm{i}t}). \tag{3.6}$$

When (3.6) is substituted into (3.1) and a few simplifications are made, we obtain for f the second-order ordinary differential equation of the Euler type

$$v^2 f'' + v f' - c^2 s f = 0. \tag{3.7}$$

(The prime denotes differentiation with respect to v.) The characteristic equation reduces to $\lambda^2 = c^2 s$ and for $s = 1$ the solution of (3.7) is

$$f = A v^c + B v^{-c}. \tag{3.8}$$

It is evident that c must be real to avoid unseemly behaviour in f. For suppose that $c = a + \mathrm{i}b$. Then the part of v^c given by the imaginary part is $\exp[bt]\,(\cos b \log x + \mathrm{i} \sin b \log x)$. The temporal part becomes infinite as time increases to infinity and the spatial part undergoes oscillations of increasing frequency as $x \longrightarrow 0$. Furthermore c must be positive to give the required convergence of $u(t, x)$ to zero as $x \longrightarrow +\infty$. Since $c > 0$, we must set B equal to zero unless $0 < c < \frac{1}{2}$ to avoid a problem at the origin.

For $c < 0$ we take the negative sign in (3.5) and use Γ_{2-} to generate the solution of (3.1). The result is the same as given above with (3.8) and the subsequent comment except that the roles of A and B are interchanged. This is in agreement with the statement of Calogero [4] that one may take $c > 0$ without loss of generality. On the other hand, if $s = -1$, the characteristic equation for (3.7) is $\lambda^2 = -c^2$ and the solution is

$$f = A v^{\mathrm{i}c} + B v^{-\mathrm{i}c}. \tag{3.9}$$

This is not an acceptable solution unless c is imaginary. However, the convergence of the wavefunction at infinity is now lost. Consequently one must conclude that the quantal problem for the Hamiltonian (1.2) is not well-posed for $s = -1$. This is a somewhat stronger statement than the conclusion reached by Calogero [4] in his concluding comments, but the outcome is essentially the same.

Henceforth we treat the case $c > 0$ only. The ground-state wavefunction is

$$u_0 = \exp\left[-\left(cx + \tfrac{1}{2}\mathrm{i}t\right)\right]x^c\,\mathrm{e}^{-cit} \tag{3.10}$$

for general values of the parameter, c. In the case that $0 < c < \frac{1}{2}$ there is the additional solution, as was already noted by Calogero [4],

$$u_0 = \exp\left[-\left(cx + \tfrac{1}{2}\mathrm{i}t\right)\right]x^{-c}\,\mathrm{e}^{-cit}. \tag{3.11}$$

(The treatment for (3.11) parallels that of (3.10) and is not detailed.) To generate further wavefunctions we construct the solution surface [9]

$$\Sigma_0 = u^{-1}(x^c\,\mathrm{e}^{-cit})\exp\left[-\left(cx + \tfrac{1}{2}\mathrm{i}t\right)\right] \tag{3.12}$$

and operate upon it with Γ_{2-}. We find that

$$\Gamma_{2-}\Sigma_0 = 2\mathrm{i}c\left(x - 1 - \frac{1}{2c}\right)\mathrm{e}^{-\mathrm{i}t}\,\Sigma_0 \tag{3.13}$$

so that the first excited state has the wavefunction

$$u_1 = \left(x - 1 - \frac{1}{2c}\right)x^c\exp\left[-cx\,\mathrm{e}^{-\mathrm{i}(\frac{3}{2}+c)t}\right]. \tag{3.14}$$

Higher states are obtained by repeated application of Γ_{2-} to the solution surface (3.12).

The energy is obtained by the action of Γ_1 on the wavefunction, i.e. as

$$\begin{aligned} &\Gamma_1 u_n = E_n u_n \\ \Leftrightarrow\quad &\mathrm{i}\frac{\partial u_n}{\partial t} = E_n u_n \\ \Rightarrow\quad &E_n = n + \tfrac{1}{2} + c. \end{aligned} \tag{3.15}$$

In the case that $0 < c < \frac{1}{2}$ the first excited state of the additional solution is

$$u_1 = \left(x + 1 - \frac{1}{2c}\right)x^{-c}\exp\left[-cx\,\mathrm{e}^{-\mathrm{i}(\frac{3}{2}-c)t}\right] \tag{3.16}$$

and in general

$$E_n = n + \tfrac{1}{2} - c. \tag{3.17}$$

The concentration of the wavefunction about the origin is evident in the closeness of the zero of (3.16) at $x = -1 + 1/(2c)$ to the origin. For this restricted range of values of the parameter, c, there would appear to exist two sequences of states with intertwined energy eigenvalues. However, the question of the self-adjoint nature of the Hamiltonian for this interval of parameter values is critical. As it has already been addressed in considerable detail by Calogero and Graffi [2] and the thrust of the present work is on the group theoretic aspects of these problems, we do not repeat their discussion. The essential point is that Calogero and Graffi report that it is necessary for $E_0 > \frac{1}{2}$ and so the second possibility summarized in (3.17) is not possible.

4. Symmetries and wavefunctions for H_1

The time-dependent Schrödinger equation corresponding to H_1 is

$$2\mathrm{i}c\frac{\partial u}{\partial t} + x^3\frac{\partial^2 u}{\partial x^2} + 3x^2\frac{\partial u}{\partial x} + \left[(1 + \rho - c^2)x - \frac{c^2}{x}\right]u = 0, \tag{4.1}$$

which corresponds to the time-independent form given by Calogero and Graffi [2, equation (17b)] and obviously we use the same ordering prescription as they used. The Lie point symmetries of (4.1) are

$$\begin{aligned}\Gamma &= \mathrm{i}\partial_t \\ \Gamma_{2\pm} &= \mathrm{e}^{\pm \mathrm{i}t}\left[\pm\mathrm{i}\partial_t + x\partial_x - \left(\frac{1}{2} \pm \frac{c}{x}\right)u\partial_u\right] \\ \Gamma_3 &= u\partial_u \\ \Gamma_f &= f(t,x)\partial_u,\end{aligned} \tag{4.2}$$

where the finite algebra of the first four symmetries is again $A_1 \oplus_s sl(2, R)$. The other comments following (3.2) apply. We note that the symmetries are independent of the parameter ρ which was introduced by Calogero and Graffi [2] to cover various possible quantization schemes for H_1. The Weyl quantization scheme corresponds to $\rho = \frac{1}{2}$.

The invariants of $\Gamma_{2\pm}$ are found from the associated Lagrange system

$$\frac{\mathrm{d}t}{\pm\mathrm{i}} = \frac{\mathrm{d}x}{x} = -\frac{\mathrm{d}u}{\left(\frac{1}{2} \pm \frac{c}{x}\right)} \tag{4.3}$$

to be

$$v = x\,\mathrm{e}^{\pm \mathrm{i}t} \qquad \text{and} \qquad w = u\exp\left[\mp\frac{1}{2}\mathrm{i}t \pm \frac{c}{x}\right]. \tag{4.4}$$

We substitute

$$u = f(x\,\mathrm{e}^{\pm\mathrm{i}t})\exp\left[\pm\frac{1}{2}\mathrm{i}t \mp \frac{c}{x}\right] \tag{4.5}$$

into (4.1) to obtain the equation

$$v^2 f'' + 3vf' + (1 + \rho - c^2)f = 0 \tag{4.6}$$

for $f(v)$.

The two solutions of (4.6) lead to two solutions of (4.5), *videlicet*

$$u_+ = x^{-1+\beta}\exp\left[\left(\mp\frac{1}{2} + \beta\right)\mathrm{i}t \mp \frac{c}{x}\right] \tag{4.7}$$

$$u_- = x^{-1-\beta}\exp\left[\left(\mp\frac{1}{2} - \beta\right)\mathrm{i}t \mp \frac{c}{x}\right], \tag{4.8}$$

where $\beta = \sqrt{c^2 - \rho}$ is necessarily real. For c real and positive proper behaviour at the origin requires that the upper sign be taken in both (4.7) and (4.8), i.e. the physically acceptable solution comes from Γ_{2+}. For proper behaviour at infinity the solution (4.8) is square integrable for all β. In the case of (4.7) this is the case if $2(-1+\beta) < -1$, i.e. $c^2 - \rho < \frac{1}{4}$.

As above the energy is obtained from the eigenvalue equation

$$\Gamma_1 u = Eu \tag{4.9}$$

and we have

$$u_{0-} = x^{-1-\beta}\exp\left[\left(-\frac{1}{2} - \beta\right)\mathrm{i}t - \frac{c}{x}\right], \qquad E_{0-} = \frac{1}{2} + \beta \tag{4.10}$$

for general β and in addition

$$u_{0+} = x^{-1+\beta}\exp\left[\left(-\frac{1}{2} + \beta\right)\mathrm{i}t - \frac{c}{x}\right], \qquad E_{0+} = \frac{1}{2} + \beta \tag{4.11}$$

in the case that $\beta < \frac{1}{2}$, i.e. $c^2 < \frac{1}{4} + \rho$.

Higher order solutions are obtained by the repeated action of Γ_{2-} on the solution surfaces corresponding to u_{0-} and to u_{0+} in the acceptable parameter range. We note that this feature of two intertwined sets of wavefunctions is not unknown since it was already observed for the Ermakov–Pinney system [9] and was reported for H_1 by Calogero and Graffi [2]. As we have already noted at the end of section 3, following their work the solution with the negative sign must be discarded since the Hamiltonian ceases to be self-adjoint.

5. Connection with the Ermakov–Pinney problem

The Newtonian equation for $H^{(s)}$ is

$$\ddot{q} = \frac{1}{2q}(\dot{q}^2 - q^2 + s). \tag{5.1}$$

We manipulate (5.1) as follows:

$$\begin{aligned} &2q\ddot{q} - \dot{q}^2 + q^2 = s \\ &q^{-\frac{1}{2}}\ddot{q} - \frac{1}{2}q^{-\frac{3}{2}}\dot{q}^2 + \frac{1}{2}q^{\frac{1}{2}} = \frac{s}{2q^{\frac{3}{2}}} \\ &\left(q^{\frac{1}{2}}\right)^{\cdot\cdot} + \frac{1}{4}q^{\frac{1}{2}} = \frac{s}{4\left(q^{\frac{1}{2}}\right)^3} \\ &\ddot{x} + \frac{1}{4}x = \frac{s/4}{x^3}, \end{aligned} \tag{5.2}$$

in which we have made the substitution $x = q^{\frac{1}{2}}$.

Hence equation (5.1) is simply a transformed version of the Ermakov–Pinney equation [7, 11] with the specific parameter values $\omega^2 = \frac{1}{4}$ and $\mu = s/4$. It is well known [5] that for the quantal problem the inequality $\mu > -\frac{1}{4}$ must be satisfied to prevent 'collapse into the origin' and this has been reflected in the results of the previous section when dealing with the quantal problem associated with $H^{(s)}$. The connection between the parameters for the Ermakov–Pinney problem and Calogero's $H^{(s)}$ reveals a facet of that problem which was not obvious in the analyses of Calogero [4] or in section 3. The Schrödinger equation for the Ermakov–Pinney problem with $\omega = 1$ is [9]

$$2\mathrm{i}\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - \left(x^2 + \frac{\mu}{x^2}\right)u = 0 \tag{5.3}$$

and the creation and annihilation symmetries are

$$Y_{2\pm} = \mathrm{e}^{\pm 2\mathrm{i}t}\left[\partial_t \pm \mathrm{i}x\partial_x \mp \mathrm{i}u\left(\tfrac{1}{2} \pm x^2\right)\partial_u\right]. \tag{5.4}$$

We obtain the ground-state wavefunction using Y_{2+} in the same fashion as in section 3. It is

$$u_0 = x^{\frac{1}{2}(1\pm\alpha)}\,\mathrm{e}^{-\frac{1}{2}x^2}\exp\left[-\mathrm{i}t\left(1 \pm \tfrac{1}{2}\alpha\right)\right], \tag{5.5}$$

where $\alpha = \sqrt{1+4\mu}$. Obviously for $\mu > 0$ only the positive sign in (5.5) can be used. However, for $-\frac{1}{4} < \mu < 0$ both solutions are feasible. We generate the wavefunctions for the higher states using the symmetry Y_{2-} and it is a simple matter to show that the energy eigenvalues are given by $E_n = 2n + 1 \pm \frac{1}{2}\alpha$. Once again the solutions with the negative values of α must be discarded for the equation to remain self-adjoint.

We note that for the Ermakov–Pinney equation the energy eigenvalues increase by even integers. In the case of $H^{(+1)}$ and H_1 the energy eigenvalues increase by integral values. To complete the connection between H_1 (1.1), $H^{(+1)}$ (1.2) and the Ermakov–Pinney problem we write the last in c-isochronous form. The Hamiltonian is

$$H^c_{\mathrm{EP}} = \frac{1}{2}\left[\frac{p^2}{c} + c\left(\omega^2 q^2 + \frac{\mu}{q^2}\right)\right]. \tag{5.6}$$

Under the canonical transformation $Q = aq^2$, $P = p/(2aq)$ the Hamiltonian of the Ermakov–Pinney problem becomes

$$H^c_{\mathrm{EP}} = \frac{1}{2}\left[4a\frac{P^2 Q}{c} + c\left(\frac{\omega^2}{a}Q + \frac{a\mu}{Q}\right)\right] \tag{5.7}$$

and this is identical to $H^{(+1)}$ when we set $a = \frac{1}{4}, \omega^2 = \frac{1}{4}$ and $\mu = 4$. (For $H^{(s)}$ we have $\mu = 4s$. However, as we have already seen that the value $s = -1$ is not admissible in the quantal mechanical problem, there is no point in maintaining the fiction that it should be considered.) (The canonical transformations between the three Hamiltonians are to be found in Calogero and Graffi [2] (6) and Calogero [4] (26).)

The Schrödinger equation for the c-isochronous form of the Ermakov–Pinney problem is

$$2\mathrm{i}c\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - c^2\left(\frac{x^2}{4} + \frac{4}{x^2}\right)u = 0 \tag{5.8}$$

and the creation and annihilation symmetries are given by

$$\Gamma_{2\pm} = \mathrm{e}^{\pm\mathrm{i}t}\left[\partial_t \pm \frac{\mathrm{i}x}{2}\partial_x - \frac{\mathrm{i}}{4}(cx^2 \pm 1)u\partial_u\right]. \tag{5.9}$$

In the same manner as we have done above we calculate

$$u_0 = x^{\frac{1}{2}\sigma}\exp\left[-\tfrac{1}{4}(cx^2 + \mathrm{i}(\sigma+1)t)\right] \tag{5.10}$$

$$E_0 = \tfrac{1}{4}(\sigma + 1) \tag{5.11}$$

$$E_n = n + \tfrac{1}{4}(\sigma + 1), \tag{5.12}$$

where $\sigma = 1 \pm \sqrt{1 + 16c^2}$. We note that the corresponding results for the ground-state energy eigenvalue for H_1 and $H^{(+1)}$ are

$$E(1)_0 = \tfrac{1}{2} + \sqrt{c^2 - \rho} \quad \text{(general normal ordering)}$$

$$E(1)_0 = \tfrac{1}{2} + \sqrt{c^2 - \tfrac{1}{2}} \quad \text{(Weyl)}$$

and

$$E_0^{(+1)} = |c| + \tfrac{1}{2}$$

respectively. This additional variation, (5.11), of the value for the ground-state energy of the quantal problem for Hamiltonians which classically have the same value adds a further illustration that the acts of making a canonical transformation of a classical Hamiltonian and the quantization of that Hamiltonian are not operations which commute [13].

Equally disturbing is the dependence of the ground-state energy on the value of the parameter c which plays no role in the Newtonian description of the motion although it does affect the value of the classical energy if this is identified with the Hamiltonian.

6. Comments

The c-isochronous Hamiltonians, (1.1) and (1.2), may be obtained from the c-isochronous Hamiltonian of the Ermakov–Pinney problem by means of successive canonical transformations. Denoting the systems, Ermakov–Pinney, $H^{(+1)}$ and H_1, by 1, 2 and 3 and their canonical coordinates likewise we have the transformations

$$q_2 = \frac{1}{4}q_1^2 \qquad p_2 = \frac{2p_1}{q_1} \tag{6.1}$$

$$q_3 = \frac{1}{q_2} \qquad p_3 = -p_2 q_2^2. \tag{6.2}$$

We note that both canonical transformations are point transformations and, as such, preserve the number of Lie point symmetries of the Newtonian equations of motion. In all cases the Lie algebra of the Noether point symmetries of the action integral is $sl(2, R)$. We have observed that the same applies to the Lie point symmetries of the respective Schrödinger equations.

It is evident that we could continue this process of devising nonlinear canonical transformations to obtain a whole variety of c-isochronous Hamiltonian systems. However, we believe that the point has been made sufficiently strongly by the examples treated here. These examples were variations, classically obtained by means of canonical transformations of the Hamiltonians, of the Ermakov–Pinney problem which is well known to be intimately connected with the simple harmonic oscillator [6].

Despite the use of a consistent ordering scheme, indeed following that used by Calogero and Graffi [2] and Calogero [3, 4], the ground-state eigenvalues of the Schrödinger equations corresponding to the three different Hamiltonians treated are different. That they depend upon the value of the parameter c, which is irrelevant in the Newtonian scheme of things, is bad enough. That they depend upon the value of the parameter c in different ways seems to be somewhat strange. The reason for this from the point of view of differential equations is that the three time-dependent Schrödinger equations must in some way not be equivalent. Yet the three equations are characterized by the possession of the same nontrivial algebra[3], $sl(2, R)$. Schrödinger equations related by point transformations preserve the algebra and the properties of the solutions. The solutions are simply expressed in terms of different variables. Evidently the ordering scheme used by Calogero and Graffi [2] and Calogero [3, 4] does not destroy the algebra, but it must introduce an inequivalent representation. If it did not, the differing results would not occur.

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[3] The trivial symmetries of homogeneity and solution symmetries are common to all linear evolution equations and so are not distinguishing.

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